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AN AVERAGING RESULT FOR l^1-SEQUENCES AND APPLICATIONS TO WEAKLY CONDITIONALLY COMPACT SETS IN L^1_X

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ABSTRACT

We establish a theorem on l^1-sequences obtained by averaging of semi-norms. The result is applied in the study of weakly conditionally compact subsets of L^1_X, where X is a Banach space.

1. An averaging result

We will denote by $(r_n)_n$ the sequence of Rademacher functions on $[0, 1]$. In the first 2 propositions, $X, \| \ \|$ will be a semi-normed space.

PROPOSITION 1. *If $x_1, \cdots, x_n$ is a finite set of elements of X and $a_1, \cdots, a_n$ reals, then*

$$\int \left\| \sum_k a_k r_k(\omega) x_k \right\| d\omega \leqq \max(|a_k|; k) . \int \left\| \sum_k r_k(\omega) x_k \right\| d\omega$$

This result is well known (see [14]) and easily seen. We omit the proof.

PROPOSITION 2. *Let $(x_n)_n$ be a bounded sequence in X and $\delta > 0$ such that $\int \| \Sigma_k a_k r_k(\omega) x_k \| d\omega \geqq \delta$ for all finite sets $a_1, \cdots, a_n$ of positive reals with $\Sigma_k a_k = 1$. Then $(x_n)_n$ has an l^1-subsequence.*

PROOF. Assume $(x_n)_n$ bounded by $M > 0$ and without l^1-sequence. Then, by Rosenthal's result [12], $(x_n)_n$ has a weak Cauchy subsequence. For convenience, we suppose $(x_n)_n$ itself w* convergent. Denote by K the dual ball of X, which is of course compact in the w*-topology. We introduce the bounded sequence $(\varphi_p)_p$

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in $C(K)$, taking $\varphi_p(x^*) = \sum_{n=p+1}^{p+d-1} |\langle x_n - x_{p+d}, x^* \rangle|$ for $x^* \in K$, where $d > \delta^{-2}M^2$ is a fixed integer. Clearly $\lim_{p\to\infty} \varphi_p = 0$ pointwise and hence weakly. Consider the subsequence $(\psi_q)_q$ of $(\varphi_p)_p$ with $\psi_q = \varphi_{q.d}$. Since the closure of the convex hull $c(\psi_q; q)$ contains the 0-function, there exists a convex combination $\sum_q \lambda_q \psi_q$ uniformly bounded by δ. Hence

$$\left\| \sum_q \sum_{n=qd+1}^{(q+1)d-1} \lambda_q (x_n - x_{(q+1)d})\, r_n(\omega) \right\| \leqq \delta \quad \text{for each } \omega \in [0,1].$$

By integration, we get

$$\begin{aligned} \delta(d-1) &\leqq \int \left\| \sum_q \sum_{n=qd+1}^{(q+1)d-1} \lambda_q r_n(\omega) x_n \right\| d\omega \\ &\leqq \delta + \int \left\| \sum_q \lambda_q \left(\sum_{n=qd+1}^{(q+1)d-1} r_n(\omega) \right) x_{(q+1)d} \right\| d\omega \\ &\leqq \delta + M\sqrt{d-1}, \end{aligned}$$

which is a contradiction.

Consider the space E of all finite real sequences and denote by e_i the i-unit vector. For $x \in E$, let $\| x \|_1 = \sum_i | x_i |$.

Assume a compact Radon probability space (M, μ) given and for each $t \in M$ a semi-norm $|\ \ |_t$ on E.

PROPOSITION 3. *Let $n \in \mathbf{N}$ be fixed and assume $| x |_t$ continuous functions of t for each $x \in [e_1, \cdots, e_n]$. Then for all $\varepsilon > 0$ there exists a finite set $(\varphi_i)_{i \in I}$ of continuous functions on M, satisfying*

(1) *The φ_i are positive and bounded by* 1,

(2) *The supports* $\operatorname{supp} \varphi_i$ *are mutually disjoint.*

(3) $\mu(\bigcup_{i \in I} [\varphi_i = 1]) > 1 - \varepsilon$.

(4) *For all $x \in [e_1, \cdots, e_n]$, we have $|\, | x |_s - | x |_t \,| \leqq \varepsilon \| x \|_1$ provided s, t are in the support of a same φ_i.*

PROOF. We show that each point $s \in M$ has a neighborhood V so that $|\, | x |_s - | x |_t \,| \leqq \varepsilon \| x \|_1$ for $t \in V$ and $x \in [e_1, \cdots, e_n]$. At this point, the proof of the proposition is straightforward and the details are left for the reader.

Clearly the functions $| e_i |_t$ of t $(1 \leqq i \leqq n)$ are bounded by some $b > 0$. Consider a finite $\varepsilon/3b$-net $\mathscr{E}$ in the unit-ball of $[e_1, \cdots, e_n]$ for the $\|\ \ \|_1$-norm. It is possible to take V such that $|\, | y |_s - | y |_t \,| \leqq \varepsilon/3$ if $y \in \mathscr{E}$ and $t \in V$. Now let $x \in [e_1, \cdots, e_n]$ with $\| x \|_1 \leqq 1$ and take $y \in \mathscr{E}$ so that $\| x - y \|_1 < \varepsilon/3b$. For $t \in V$, we get

$$\big|\,|x|_s - |x|_t\big| \leqq |x-y|_s + |x-y|_t + \big|\,|y|_s - |y|_t\big|$$

$$\leqq \sum_i |x_i - y_i|\,|e_i|_s + \sum_i |x_i - y_i|\,|e_i|_t + \frac{\varepsilon}{3}$$

$$\leqq 2b\,\|x-y\|_1 + \frac{\varepsilon}{3}$$

$$< \varepsilon.$$

Thus the required property is satisfied.

THEOREM 4. *Assume* $\sup_{i,t} |e_i|_t < \infty$ *and* $|x|_t$ *a measurable function of* t *for all* $x \in E$. *Clearly* $|x| = \int |x|_t \,.\, \mu(dt)$ *defines a semi-norm on* E.

Suppose that $(e_i)_i$ *is an* l^1*-sequence for* $|\ \ |$. *Then there exist* $t \in M$ *and a subsequence of* $(e_i)_i$ *which is an* l^1*-sequence for* $|\ \ |_t$.

We will first prove the theorem assuming the $|x|_t$ continuous functions of t. For convenience, suppose $|e_i|_t$ bounded by 1. Proposition 3 yields us for each $i \in \mathbf{N}$ a finite set $(\varphi_{ij})_{j \in I_i}$ of continuous functions on M, satisfying

(1) The φ_{ij} are positive and bounded by 1.

For each $i \in \mathrm{N}$

(2) The supports $\operatorname{supp} \varphi_{ij}$ are mutually disjoint.

(3) $\mu(M_i) > 1 - 2^{-i}$ where $M_i = \bigcup_{j \in I_i} [\varphi_{ij} = 1]$.

(4) $\big|\,|x|_s - |x|_t\big| \leqq 2^{-i}\|x\|_1$ if $x \in [e_1, \cdots, e_i]$ and s, t are in the support of a same φ_{ij}.

Let us now introduce the real vector space $\mathscr{X}$ spanned by the unit vectors $(e_{ij})_{i \in \mathrm{N}, j \in I_i}$. If we take

$$\Big|\Big|\Big| \sum_i{}' \sum_{j \in I_i} a_{ij} l_{ij} \Big|\Big|\Big| = \sup_t \int \Big| \sum_i{}' \sum_{j \in I_i} a_{ij} \varphi_{ij}(t) r_i(\omega) e_i \Big|_t \, d\omega,$$

then $|||\ \ |||$ is a semi-norm on $\mathscr{X}$. Remark that $|||\, e_{ij}\, ||| \leqq 1$. Defining $F_i(t) = \sum_{j \in I_i} \varphi_{ij}(t) e_{ij}$, we obtain a bounded by 1 sequence $(F_i)_i$ of continuous $\mathscr{X}$-valued functions on M.

LEMMA 5. *Let* $i_0 \in \mathbf{N}$, $(a_i)_i$ *a sequence of reals which are* 0 *except for finitely many* $i \geqq i_0$ *and* $s \in M$. *Then*

$$\Big(\inf_{i \geqq i_0} \sum_{j \in I_i} \varphi_{ij}(s)^2\Big) \cdot \int \Big| \sum_i a_i r_i(\omega) e_i \Big|_s \, d\omega \leqq \Big|\Big|\Big| \sum_i a_i F_i(s) \Big|\Big|\Big|$$

$$\leqq \int \Big| \sum_i a_i r_i(\omega) e_i \Big|_s \, d\omega + 2^{-i_0} \sum_i |a_i|.$$

PROOF. If we take $\psi_i(s,t)=\sum_{j\in I_i}\varphi_{ij}(s)\varphi_{ij}(t)$, then we find that

$$\left|\left|\left|\sum_i a_iF_i(s)\right|\right|\right| = \sup_t \int \left|\sum_i \psi_i(s,t)a_ir_i(\omega)e_i\right|_t d\omega,$$

by definition of $|||\ \ |||$. The first inequality is obtained by taking $t=s$, using Proposition 1. To obtain the second inequality, let us fix $t\in M$. If $i_1=\max\{i\geqq i_0;\ \psi_i(s,t)\neq 0\}$, then $s,t\in\operatorname{supp}\varphi_{i_1j}$ for some $j\in I_{i_1}$. Therefore, by construction of the φ_{ij}, it follows that

$$\left|\sum_i \psi_i(s,t)a_ir_i(\omega)e_i\right|_t \leqq \left|\sum_{i=i_0}^{i_1}\psi_i(s,t)a_ir_i(\omega)e_i\right|_s + 2^{-i_1}\sum_{i=i_0}^{i_1}\psi_i(s,t)\,|\,a_i\,| \quad \text{for each } \omega\in[0,1].$$

Since $\psi_i(s,t)\leqq 1$, integration yields us

$$\int \left|\sum_i \psi_i(s,t)a_ir_i(\omega)e_i\right|_t d\omega \leqq \int \left|\sum_i a_ir_i(\omega)e_i\right|_s d\omega + 2^{-i_0}\sum_i |\,a_i\,|$$

again by Proposition 1. This completes the proof.

LEMMA 6. *There is some $t\in M$ such that $(F_i(t))_i$ is not a weakly 0 sequence in $\mathscr{X}$.*

PROOF. Assume $\lim_i F_i(t)=0$ weakly in $\mathscr{X}$ for each $t\in M$. Denote by K the dual ball of $\mathscr{X}$ equipped with the w*-topology. For each $i\in\mathbf{N}$, consider the function G_i on the compact space $M\times K$, given by $G_i(t,x^*)=\langle F_i(t),x^*\rangle$. It is easily seen that G_i is continuous and bounded by 1. Since $(G_i)_i$ is pointwise converging to 0, $\lim_i G_i=0$ weakly in $C(M\times K)$.

Choose $\iota>0$ arbitrarily small and $i_0\in\mathbf{N}$ with $2^{-i_0+2}<\iota$. There is a convex combination $\sum_{i\geqq i_0}\lambda_iG_i$ which is uniformly bounded by $\iota/2$. Hence $|||\sum_{i\geqq i_0}\lambda_iF_i(t)|||\leqq\iota/2$ for all $t\in M$. Take $N=\bigcap_{i\geqq i_0}M_i$ and remark that $\mu(N)\geqq 1-\sum_{i\geqq i_0}2^{-i}>1-\iota/2$. We now use Lemma 5 and obtain

$$\int \left|\sum_i \lambda_ir_i(\omega)e_i\right|_t d\omega \leqq \frac{\iota}{2} \qquad \text{if } t\in N.$$

Hence, by the Fubini theorem,

$$\int \left|\sum_i \lambda_ir_i(\omega)e_i\right| d\omega = \int\left(\int \left|\sum_i \lambda_ir_i(\omega)e_i\right|_t d\omega\right)dt \leqq \frac{\iota}{2}+\mu(M\backslash N)\leqq\iota.$$

Taking $\iota>0$ sufficiently small, we get a contradiction on the hypothesis that $(e_i)_i$ is an l^1-sequence for $|\ \ |$.

Proof of the theorem in the continuous case

By Lemma 6, there is some $t \in M$ such that $(F_i(t))_i$ does not converge weakly to 0 in $\mathscr{X}$. Hence there exist $\delta > 0$ and a subsequence $(F_{i_k}(t))_k$ so that $\mathrm{dis}_{|||\ |||}(c(F_{i_k}(t), k), 0) \geqq \delta$.

Take k_0 with $2^{-k_0+1} < \delta$.

If $(a_k)_k$ is a sequence of positive reals which are 0 except for finitely many $k \geqq k_0$ and $\Sigma_k a_k = 1$, then we deduce from Lemma 5

$$\delta \leqq \left|\left|\left| \sum_k a_k F_{i_k}(t) \right|\right|\right| \leqq \int \left| \sum_k a_k r_{i_k}(\omega) e_{i_k} \right|_t d\omega + 2^{-k_0}$$

and thus $\int |\Sigma_k a_k r_{i_k}(\omega) e_{i_k}|_t d\omega \geqq \delta/2$. It follows from Proposition 2 that $(e_{i_k})_{k \geqq k_0}$ has an l^1-subsequence for $|\ \ |_t$.

Proof of the theorem in the general case

Suppose $\delta > 0$ such that $|x| \geqq \delta \|x\|_1$ for $x \in E$. Consider a sequence $\mathscr{E}$ in E which is dense for $\|\ \ \|_1$. By Lusin's theorem, there exists a compact subset M' of M so that $\mu(M') > 1 - \delta/2$ and the mappings $|y|_t$ on M' are continuous for each $y \in \mathscr{E}$. It is routine to verify that also $|x|_t$ is continuous on M' for each $x \in E$. Define the semi-norm $|\ \ |'$ on E by $|x|' = \int_{M'} |x|_t \mu(dt)$. Since for $x \in E$, we have

$$|x|' \geqq |x| - \int_{M \setminus M'} |x|_t \mu(dt) \geqq \delta \|x\|_1 - \mu(M \setminus M') \|x\|_1 \geqq \tfrac{1}{2}\delta \|x\|_1,$$

we reduced the problem to the continuous case.

2. Weakly conditionally compact subsets of L^1_X

We say that a subset A of a Banach space Y is weakly conditionally compact provided every sequence in A has a weak Cauchy subsequence.

We can then reformulate Rosenthal's result [12] by saying that if A is bounded then either A is weakly conditionally compact or A contains an l^1-sequence.

In this section, (M, μ) will be a fixed compact Radon probability space and X a fixed Banach space. If $1 \leqq p < \infty$, then $L^p_X = L^p_X(M, \mu)$ will denote the space of p-integrable X-valued function classes on (M, μ) (cf. [6]).

A set A of L^1_X is said to be equi-integrable provided $\{\|f\|; f \in A\}$ is equi-integrable. The following property is well known [5].

PROPOSITION 7. *Every weakly conditionally compact subset of L^1_X is equi-integrable.*

The next result was shown by B. Maurey and G. Pisier (see [11]) and also independently by the author. It will be obtained here as a corollary of the stronger result established in the preceding section.

THEOREM 8. *Let $(f_n)_n$ be an equi-integrable sequence in L^1_X and assume $(f_n(t))_n$ without l^1-sequence for almost all $t \in M$. Then $\{f_n;\, n\}$ is weakly conditionally compact in L^1_X.*

PROOF. Assume $(f_n)_n$ a uniformly bounded sequence of X-valued μ-measurable functions on M and $\delta > 0$, such that

$$\int \left\| \sum_n{}' a_n f_n(t) \right\| \mu(dt) \geqq \delta \sum_n{}' |a_n|$$

for every finitely supported sequence $(a_n)_n$ of scalars. The only thing to prove is that then $(f_n(t))_n$ has an l^1-sequence for some $t \in M$.

Let E be as in the first section and introduce the semi-norm $|\ \ |_t$ on E by $|x|_t = \|\Sigma_n x_n f_n(t)\|$ for $x \in E$. The conditions of Theorem 4 are satisfied and we get the required conclusion.

Combining Proposition 7 and Theorem 8, we get

COROLLARY 9. *Let X be a Banach space which does not contain l^1 (isomorphically). Then a subset A of L^1_X is weakly conditionally compact if and only if it is equi-integrable.*

We have the following stability result:

PROPOSITION 10. *Let A be a weakly conditionally compact subset of L^1_X and B a bounded subset of $L^\infty_{\mathbf{R}}$. Then the set $A \cdot B = \{f \cdot \varphi;\, f \in A, \varphi \in B\}$ is also weakly conditionally compact in L^1_X.*

PROOF. We may assume B the unit ball of $L^\infty_{\mathbf{R}}$. Suppose the statement untrue, then there would be a sequence $(f_n)_n$ in A and a sequence $(\varphi_n)_n$ in B such that $(f_n \cdot \varphi_n)_n$ is an l^1-basis, i.e., for some $\delta > 0$

$$\left\| \sum_n{}' a_n f_n \cdot \varphi_n \right\|_1 \geqq \delta \sum_n{}' |a_n|$$

for all finitely supported sequences $(a_n)_n$ of scalars. Hence, by Proposition 1, also

$$\int \left\| \sum_n{}' a_n r_n(\omega) f_n \right\|_1 d\omega \geqq \int \int \left\| \sum_n{}' a_n r_n(\omega) f_n(t) \varphi_n(t) \right\| d\omega\, \mu(dt)$$

$$\geqq \delta \sum_n{}' |a_n|.$$

We apply Proposition 2 to obtain an l^1-subsequence of $(f_n)_n$. But this contradicts the hypothesis on A.

We now present another proof of a result of G. Pisier [11].

THEOREM 11. *If $1 < p < \infty$, then l^1 imbeds in L^p_X if and only if l^1 imbeds in X.*

PROOF. Fix $1 < p < \infty$ and remark that $L^p_{\mathbf{R}}$ is a uniformly convex Banach space. Let $0 < \varepsilon < \frac{1}{4}$ be choosen and take $0 < \delta < \frac{1}{4}$ such that if $\| \varphi \|_p = \| \psi \|_p = 1$ and $\| \varphi - \psi \|_p > \varepsilon$ then $\| \varphi + \psi \|_p < 2(1-\delta)$ for all φ, ψ in $L^p_{\mathbf{R}}$. If l^1 imbeds in L^p_X, then there is a sequence $(f_n)_n$ in the unit sphere of L^p_X, satisfying

$$\left\| \sum_n{}' a_n f_n \right\|_p \geqq (1-\delta) \sum_n{}' | a_n |$$

for all finitely supported sequences $(a_n)_n$ of reals (using the James regularization property [7]).

For each f_n, there is some g_n in L^∞_X and φ_n in $L^p_{\mathbf{R}}$ so that $\| g_n \|_\infty = 1$, $\varphi_n = \| f_n \|$ and $f_n = g_n \cdot \varphi_n$. If $m \neq n$, we get $\| \varphi_m + \varphi_n \|_p \geqq \| f_m + f_n \|_p \geqq 2(1-\delta)$ and therefore $\| \varphi_m - \varphi_n \|_p \leqq \varepsilon$. Consider a function φ in $L^\infty_{\mathbf{R}}$ with $\| \varphi - \varphi_1 \|_p < \varepsilon$ and remark that $\| f_n - g_n \cdot \varphi \|_p \leqq \| \varphi_n - \varphi_1 \|_p + \| \varphi_1 - \varphi \|_p < 2\varepsilon$. Hence, we see that

$$\| \varphi \|_\infty \left\| \sum_n{}' a_n g_n \right\|_1^{1/p} \geqq \left\| \sum_n{}' a_n g_n \cdot \varphi \right\|_p \geqq 1 - \delta - 2\varepsilon > \frac{1}{4}$$

if $(a_n)_n$ is a sequence of reals with $\Sigma_n | a_n | = 1$. Therefore $(g_n)_n$ is an l^1-sequence in L^1_X and l^1 imbeds in X by Corollary 9.

It is a natural problem to characterize weakly conditionally compact subsets of L^1_X. We already solved the problem if X has no l^1-subspace (Corollary 9).

PROPOSITION 12. *For a subset A of L^1_X, the following properties are equivalent:*

(1) *For every $\varepsilon > 0$ there exists a weakly conditionally compact subset W of X such that for each function $f \in A$ there is a measurable subset M_f of M satisfying*

(i) $\int_{M_f} \| f(t) \| \mu(dt) < \varepsilon$,

(ii) $f(M \backslash M_f) \subset W$.

(2) *The same as* (1) *but with W replaced by $W + B(0, \varepsilon)$ in* (ii).

PROOF. Suppose that A satisfied (2) and fix $\varepsilon > 0$. Consider for each n a subset W_n of X associated with the positive number $\varepsilon_n = \varepsilon \cdot 2^{-n}$. It is easily checked that $W = \bigcap_n (W_n + B(0, \varepsilon_n))$ is still weakly conditionally compact. For fixed f in A let for each n the set M^n_f be such that $\int_{M^n_f} \| f(t) \| \mu(dt) < \varepsilon_n$ and $f(M \backslash M^n_f) \subset W_n + B(0, \varepsilon_n)$. Take $M_f = \bigcup_n M^n_f$, for which (i) and (ii) of (1) are clearly true.

We agree to say that a set A in L^1_X has property (U) provided it fulfils condition (1) of Proposition 12.

PROPOSITION 13. *If the subset A of L^1_X verifies* (U), *then*

(1) *A is weakly conditionally compact.*

(2) *Given for each $f \in A$ an automorphism θ_f of the measure space (K, μ), then also the set $\{f \circ \theta_f;\, f \in A\}$ has property* (U).

PROOF. (1) If A is not weakly conditionally compact, there is a sequence $(f_n)_n$ of functions in A and $\varepsilon > 0$ such that if $(g_n)_n$ is a sequence in L^1_X with $\|f_n - g_n\|_1 < \varepsilon$ then also $(g_n)_n$ is not weakly conditionally compact. Take W and for all $f \in A$ a subset M_f of M satisfying (1) of Proposition 12. Then the sequence $(g_n)_n$ in L^1_X with $g_n = f_n \cdot \chi_{M \setminus M_n}$ $(M_n = M_{f_n})$ is not weakly conditionally compact. Since A is certainly equi-integrable, $(g_n)_n$ is equi-integrable. We apply Theorem 8 and get some $t \in M$ for which $(g_n(t))_n$ has an l^1-subsequence. But $\{g_n(t);\, n\} \subset \{0\} \cup W$, contradicting the hypothesis on W.

(2) Follows readily from the definition of property (U).

THEOREM 14. *For subsets A of L^1_X, property* (U) *is equivalent to weakly conditionally compactness in the following cases:*

(1) *X is a Banach space without l^1-subspace.*

(2) *X is an $L^1(N, \nu)$-space, where (N, ν) is a positive σ-finite measure space.*

PROOF. (1) Is equivalent to Corollary 9.

(2) If $X = L^1(N, \nu)$, the space L^1_X can be identified with the space $L^1(M \times N) = L^1(M \times N, \mu \otimes \nu\}$, by the Fubini theorem. For $f \in L^1_X$, $\hat{f}$ will denote some member of the corresponding function class in $L^1(M \times N)$. The function $\hat{f}$ will moreover be assumed measurable for the product σ-algebra.

If A is a weakly conditionally compact subset of L^1_X, then A is relatively weakly compact viewed as a set in $L^1(M \times N)$, which is weakly sequentially complete.

Fix now $\varepsilon > 0$. Let $0 < \delta < \varepsilon/2$ be such that $\int_{M'} \|f(t)\|_1 \, \mu(dt) < \varepsilon$ for all $f \in A$ whenever $\mu(M') < \delta$. By a well known result (cf. [6], III), there exist some measurable subset N' of N and some $a > 0$ satisfying $\nu(N \setminus N') < \infty$ and for each $f \in A$

$$\int_{P_f} |\hat{f}(t, u)| \, \mu(dt) \otimes \nu(du) < \frac{\delta^2}{2}, \quad \text{where}$$

$$P_f = \{(t, u) \in M \times N;\, |\hat{f}(t, u)| > a\},$$

$$\int_{M \times N'} |\hat{f}(t, u)| \mu(dt) \otimes v(du) < \frac{\delta^2}{2}.$$

The subset $W = \{\varphi \in L^1(N, \nu);\ \|\varphi\|_\infty \leqq a$ and $\varphi = 0$ on $N'\}$ of $L^1(N, \nu)$ is relatively weakly compact. Fix now some $f \in A$. Take

$$M'_f = \left\{ t \in M;\ \int_{P_f(t)} |\hat{f}(t, u)|\, \nu(du) > \delta \right\},$$

$$M''_f = \left\{ t \in M;\ \int_{N'} |\hat{f}(t, u)|\, \nu(du) > \delta \right\}$$

and

$$M_f = M'_f \cup M''_f.$$

The reader will verify that $\mu(M'_f) < \delta/2$, $\mu(M''_f) < \delta/2$ and hence $\mu(M_f) < \delta$. Therefore $\int_{M_f} \|f(t)\|_1\, \mu(dt) < \varepsilon$.

For $t \in M \setminus M_f$, consider the function g on N, which is 0 on $P_f(t) \cup N'$ and $\hat{f}_t$ elsewhere. Then the g-function class belongs to W. Furthermore

$$\int |\hat{f}_t(u) - g(u)|\nu(du) \leqq \int_{P_f(t)} |\hat{f}(t, u)|\, \nu(du) + \int_{N'} |\hat{f}(t, u)|\, \nu(du) \leqq 2\delta < \varepsilon.$$

By Proposition 12, A satisfies (U).

In general, it is untrue that weakly conditionally compactness implies (U). We will illustrate this with an example. Let K be the Cantor set $\{0, 1\}^{\mathbf{N}}$ and μ the Haar measure. For each n, consider the measure automorphism θ_n of K obtained by permutation of the n-coordinate. (θ_n^2 is the identity). Take $\mathscr{C} = \bigcup_n \{0, 1\}^n$, consisting of the finite complexes. For $\varepsilon, \varepsilon' \in \mathscr{C}$, we write $\varepsilon < \varepsilon'$, provided ε' starts with ε. This yields us a partial order on $\mathscr{C}$. We say that $\varepsilon, \varepsilon' \in \mathscr{C}$ are comparable if either $\varepsilon < \varepsilon'$ or $\varepsilon' < \varepsilon$ and otherwise they are called incomparable. If $\nu \in K$, then $\nu \mid n \in \mathscr{C}$ will consist of the first n elements of ν. Remark that the members of the set $\{\theta_n(\nu) \mid n;\ n\}$ are mutually incomparable for all $\nu \in K$. X will be the Banach space obtained by completion of the linear space spanned by unit vectors $(e_\varepsilon)_{\varepsilon \in \mathscr{C}}$ under the norm

$$\|x\| = \sup_{\varepsilon_1, \cdots, \varepsilon_n} \sum_{k=1}^{n} \left(\sum_{\varepsilon_k < \varepsilon'} |x_{\varepsilon'}|^2 \right)^{\frac{1}{2}}$$

where the supremum is taken over all finite sets $\varepsilon_1, \cdots, \varepsilon_n$ of mutually incomparable members of $\mathscr{C}$.

It is clear that if $\nu \in K$, then $(e_{\nu|n})_n$ is isometrically equivalent with the l^2-basis. It follows from the preceding remark that $(e_{\theta_n(\nu)|n})_n$ is isometrically equivalent with the l^1-basis. We introduce the sequence $(f_n)_n$ in L^1_X, by taking $f_n(\nu) = e_{\nu|n}$ for each $\nu \in K$. The following facts are straightforward:

(1) $(f_n)_n$ is an l^2-basis in L^1_X.

(2) $(f_n \circ \theta_n)_n$ is an l^1-basis in L^1_X.

By (1), the set $\{f_n; n\}$ is relatively weakly compact and hence certainly weakly conditionally compact in L^1_X. But $\{f_n; n\}$ fails property (U), by (2) and Proposition 13.

We end with the following two interesting problems, which seem to be unsolved so far:

(1) Is it true that L^1_X is weakly sequentially complete if X is weakly sequentially complete?

(2) The same question for the Dunford–Pettis property.

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